EQUILIBRIUM AND STABILITY OF A CYLINDRICAL LINEAR PINCH WITH A HOMOGENEOUS TEMPERATURE DISTRIBUTION

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The equilibrium and stability of a high-current discharge have been theoretically investigated in a dense optically gray plasma. The plasma is assumed to be completely opaque to long-wave photons and completely transparent to short-wave photons. The threshold frequency is determined by setting the diameter of the plasma pinch equal to the mean free path of the photons. We solve the equations of magnetohydrodynamics together with the equation of radiative transfer. We show that in a gray plasma an equilibrium state can exist with a practically homogeneous temperature distribution over the discharge cross section. Temperature homogeneity is ensured by the large radiant thermal conductivity, which is related to the long-wave radiation. The radiant thermal conductivity also causes the discharge to be stable with respect to superheating. We analyze the possibility of using such a discharge for the energy pumping of a laser. We show that for discharge currents of order 10^6 A, the efficiency of a gray discharge exceeds the efficiency of an opaque discharge by a factor of three.

The equilibrium and stability of a high-current pinched discharge have been investigated in [1-6] for the limiting cases of optically opaque and completely transparent plasmas. It was shown that for high radiant thermal conductivity, the plasma temperature in the discharge, to a high degree of accuracy, is homogeneous over the discharge cross section if the discharge current does not exceed some maximal value J_{max}. Such a plasma radiates as an ideal black body, i.e., it ensures maximum radiation output in any given spectral range. In order to satisfy the conditions of applicability for the approximation of radiant thermal conductivity, the discharge current must be greater than some minimal value J_{min} ($J_{max}/J_{min} \approx 3$).

It would appear that such a discharge is optimal from the point of view of its use as a source of radiation for pumping a laser, especially since it is more stable than a transparent discharge, which is subject to a rapidly developing superheat instability. It is evident, however, that an optically opaque discharge has low efficiency. Actually, for laser pumping we usually must ensure a radiation maximum in some spectral region. We were particularly interested in the region 2000-3000 Å. For a black body, the radiation in this range for T = 3-5 eV is ≈6% of the total radiation. A transparent discharge in the atmosphere of a number of elements has the necessary selectivity of radiation, but, as has already been said, is much less stable. Thus, the radiator that is optimal from the point of view of its use as a source of radiation for pumping a laser is a "gray" radiator close to black in the given wavelength range, and transparent for radiation of shorter wavelength. We can expect that the long-wave radiation will ensure the value of radiant thermal conductivity that is necessary to eliminate the superheat instability.

1. Formulation of the Problem and Fundamental Equations

The complete system of equations of magnetohydrodynamics taking account of the radiant energy flux for a completely ionized plasma, which can be assumed to be an ideal gas, can be written in the form [7, 8]

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$$\operatorname{div}\mathbf{B}=0,\quad \operatorname{rot}\mathbf{B}=\frac{4\pi}{c}\mathbf{j}=\frac{4\pi}{c}\sigma\left(\mathbf{E}+\frac{1}{c}\mathbf{v}\times\mathbf{B}\right),$$
$$-c\operatorname{rot}\mathbf{E}=\frac{\partial\mathbf{B}}{\partial t}=\operatorname{rot}(\mathbf{v}\times\mathbf{B})-\frac{c^2}{4\pi}\operatorname{rot}\left(\frac{1}{\sigma}\operatorname{rot}\mathbf{B}\right),$$
$$\rho\left(\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}\right)=-\nabla P+\frac{1}{4\pi}(\operatorname{rot}\mathbf{B}\times\mathbf{B}),\qquad \frac{\partial\rho}{\partial t}+\operatorname{div}\rho\mathbf{v}=0,$$
$$\frac{\partial}{\partial t}\left(\rho\frac{v^2}{2}+\rho\varepsilon+\frac{B^2}{8\pi}\right)+\operatorname{div}(\mathbf{q}+\mathbf{S})=0,\tag{1.1}$$
$$P=\frac{(1+z)k\rho T}{M}\equiv v_s^2\rho,\qquad \sigma=\frac{\alpha}{z}T^{3/2},$$
$$\varepsilon=c_VT=\frac{3}{2}\frac{(1+z)kT}{M}\equiv\frac{3}{2}v_s^2,$$
$$\mathbf{q}=\rho\mathbf{v}\left(\frac{v^2}{2}+\varepsilon+\frac{P}{\rho}\right)+\frac{1}{4\pi}\mathbf{B}\times(\mathbf{v}\times\mathbf{B})+\frac{c^2}{16\pi^2\sigma}\operatorname{rot}\mathbf{B}\times\mathbf{B}.$$

Here $\mathbf{q}$ is the energy flux of the substance; $\mathbf{S}$ is the radiant flux, which will be determined below; M is the ion mass; z is the ion effective charge; v_s is the velocity of isothermal sound; σ is the conductivity of the medium being considered; and $\alpha = 4 \cdot 10^7$. In writing down system (1.1), in the same way as was done in [1-6], we neglect the electronic thermal conductivity in comparison with radiative heat exchange and the viscosity terms and we also neglect the radiant energy in comparison with the internal (thermal) energies of the plasma particles.

To determine $\mathbf{S}$ we must solve simultaneously the equation of radiative transfer and the system (1.1). However, such a problem is mathematically extremely complicated. For a gray body, as the simplest approximation, we can divide the radiant flux into two parts: $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$, where $\mathbf{S}_1$ is the radiant flux of the long-wave photons with frequencies $\nu < \nu_0$, for which the medium is completely opaque, and $\mathbf{S}_2$ is the radiant flux of the short-wave photons with frequencies $\nu > \nu_0$, for which the medium is transparent. Here ν_0 is some threshold frequency which is determined from the equation

$$\tau=\varkappa_\nu'(\nu_0)r_p=1,\tag{1.2}$$

where τ is the optical thickness of the discharge; $\varkappa_\nu'$ is the spectal coefficient of absorption taking account of re-emission (see [8]), and r_p is the equilibrium dimension (radius) of the plasma cylinder, which will be determined below.

We calculate the fluxes $\mathbf{S}_1$ and $\mathbf{S}_2$ for the case of Bremsstrahlung, when [8]

$$\varkappa_\nu'=\frac{4.1\cdot10^{-23}z^3N^2}{T^{7/2}}\frac{1-e^{-x}}{x^3}\qquad\left(x=\frac{h\nu}{kT}\right).\tag{1.3}$$

For $\mathbf{S}_1$, according to [1] we have

$$\mathrm{S}_1=\int_0^{\nu_0}d\nu\mathrm{S}_\nu=-\frac{4\pi}{3}\frac{kT}{h}\int_0^{x_0}\frac{\nabla I_{\nu p}}{\varkappa_\nu'}dx=-\frac{\beta_0T^{13/2}f(x_0)}{N^2z^3}\nabla T,\tag{1.4}$$
$$I_{\nu p}=\frac{2(kT)^3}{c^2h^3}\frac{x^3}{e^x-1},\quad f(x_0)=\int_0^{x_0}\frac{e^{2x}x^7}{(e^x-1)^3}.$$

Here $I_{\nu p}$ is the intensity of radiation of an absolute black body and $\beta_0 = 2.8 \cdot 10^{17}$. For $\mathbf{S}_2$, according to [2], we obtain

$$\operatorname{div}\mathrm{S}_2\equiv\frac{1}{r}\frac{\partial}{\partial r}(rS_2)=\int_{\nu_0}^{\infty}d\nu\int d\Omega\varkappa_\nu'I_{\nu p}=\gamma_0N^2\sqrt{T}z^3e^{-x_0},\qquad \gamma_0=1.4\cdot10^{-27}.\tag{1.5}$$

2. Equilibrium of a Punched Discharge in an Optically Gray Plasma

We consider the stationary equilibrium state of a simple cylindrical discharge in a gray plasma when the temperature is homogeneous over the cross section. We see from system (1.1) that in equilibrium the electric field $\mathbf{E}_0$, which produces a current in the plasma, can be assumed to be homogeneous over the cross section. Then the spatial distribution of pressure, of density, and of magnetic field can easily be determined from the first three equations of system (1.1) in the same way as was done in [1]:

$$B_0 = \frac{2\pi}{c} j_0 r = \frac{2J}{c r_p^2} r = \sqrt{4\pi P_0(0)} \frac{r}{r_p},$$

$$P_0 = P_0(0)\left(1 - \frac{r^2}{r_p^2}\right), \qquad \rho_0 = \rho_0(0)\left(1 - \frac{r^2}{r_p^2}\right), \tag{2.1}$$

$$r_p^2 = \frac{c^2 P_0(0)}{\pi j_0^2} = \frac{J^2}{\pi c^2 P_0(0)}, \qquad J = \pi r_p^2 \sigma_0 E_0,$$

where $P_0(0)$, $T_0(0)$, and $\rho_0(0)$ are the values of the pressure, temperature, and density on the discharge axis, and J is the given total current in the discharge. We also assume given the total number of particles in the discharge per unit length N_n. Then

$$N_n = \frac{\pi r_p^2}{2} N_0(0) = \frac{J^2}{2kc^2(1+z)T_0(0)}, \quad T_0(0) = \frac{J^2}{2kc^2(1+z)N_n}. \tag{2.2}$$

Next substituting Eqs. (2.1), (1.4), and (1.5) into the heat-balance equation of system (1.1), we find the temperature distribution $T_0(r)$ over the cross section of the plasma cylinder:

$$T_0(r) = T_0(0)\left\{1 + \frac{A_0 r^2}{4C_0 T_0(0)}\left(1 - \frac{3r^2}{2r_p^2} + \frac{10r^4}{9r_p^4} - \frac{5r^6}{12r_p^6} + \frac{r^8}{15r_p^8}\right) - \frac{D_0 r^2}{2C_0 T_0(0)}\left(1 - \frac{r^2}{r_p^2} + \frac{r^4}{3r_p^4}\right)\right\}. \tag{2.3}$$

$$A_0 = \gamma_0 N_0^2(0)\sqrt{T_0(0)}\, z^3 e^{-x_0}, \quad C_0 = \frac{\beta_0 T_0^{13/2}}{z^3 N_0^2(0)} f(x_0), \quad D_0 = \frac{P_0(0)c^2}{2\pi\sigma_0 r_p^2}.$$

We can now write the condition of applicability of the approximation assumed above to the homogeneity of the plasma temperature over the cross section of the discharge:

$$A_0 r_p^2, \quad 2D_0 r_p^2 \ll 4C_0 T_0(0). \tag{2.4}$$

To completely close the problem, we write the energy balance on the discharge surface:

$$S(r_p) = \frac{r_p}{2}\sigma_0 E_0^2 = \frac{J^2}{2\pi^2\sigma_0 r_p^3} = \frac{J^2 z}{2\pi^2 \alpha T_0^{3/2} r_p^3}. \tag{2.5}$$

As was shown in [9], the radiant flux from the surface of a plasma cylinder with a homogeneous temperature distribution is given by the equation

$$S = \frac{2\pi(kT_0)^4}{c^2h^3}\int_0^\infty \frac{x^3\,dx}{e^x - 1}\left\{1 - \frac{4}{\pi}\int_1^\infty \frac{dy}{y^3\sqrt{y^2-1}}\int_0^1 \frac{\mu\,d\mu}{\sqrt{1-\mu^2}}\exp\left(-\frac{16}{15} y\mu^5 \varkappa_\nu' r_p\right)\right\}. \tag{2.6}$$

This integral can be calculated only approximately; therefore, we divide it into two, assuming that in the long-wave part from 0 to x_0 the discharge radiates as a black body, i.e., $\tau = \varkappa_\nu' r_p \gg 1$, and for $x > x_0$ the discharge is transparent, i.e., $\tau \ll 1$. As a result, Eq. (2.6) can be written in the form

$$S \approx \frac{2\pi(kT_0)^4}{c^2h^3}\left\{\int_0^{x_0} \frac{x^3\,dx}{e^x - 1} + \int_{x_0}^\infty \frac{x^3\,dx}{e^x - 1}\left[1 - \frac{4}{\pi}\int_1^\infty \frac{dy}{y^3\sqrt{y^2-1}}\int_0^1 \frac{\mu\,d\mu}{\sqrt{1-\mu^2}}\left(1 - \frac{16}{15} y\mu^5 \varkappa_\nu' r_p\right)\right]\right\}. \tag{2.7}$$

Here the first term in the braces describes the radiation of long-wave photons, and in the limit $x_0 \to \infty$ it gives the well-known expression σT^4, where σ is the Stefan–Boltzmann constant. The second term describes the radiation of the short-wave photons where, in the derivation, the exponent is expanded in a series, which is possible since, in this region, $\varkappa_\nu' r_p \ll 1$, $\mu \le 1$, and the dy portion of the integral gives the region of y values close to unity. Integrating (2.7), after simple transformations we obtain

$$S = \frac{2\pi(kT_0)^4}{c^2h^3}\left(\varphi(x_0) + \frac{2}{3}\psi(x_0)\right) \equiv \frac{2\pi(kT_0)^4}{c^2h^3}\xi(x_0),$$

$$\varphi(x_0) = \int_0^{x_0} \frac{x^3\,dx}{e^x - 1}, \qquad \psi(x_0) = \frac{x_0^3}{e^{x_0} - 1}. \tag{2.8}$$

We can now calculate the equilibrium radius of the discharge r_p, substituting the second of Eqs. (2.2) for T_0 and the expression obtained for S into Eq. (2.5):

$$r_p^3 = \frac{6.7 \cdot 10^{25}}{\xi(x_0)} z(1+z)^{11/2} \frac{N_n^{11/2}}{J^9} \tag{2.9}$$

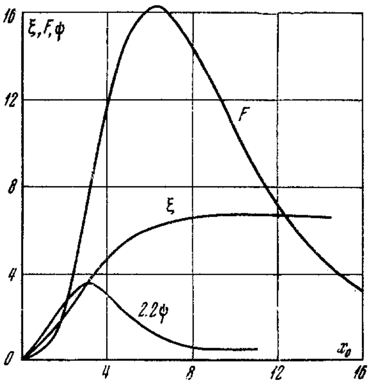

Fig. 1

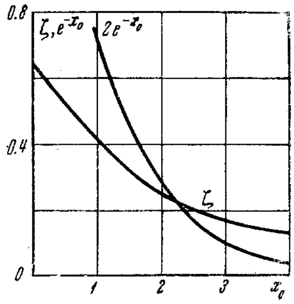

Fig. 2

Taking account of this equation, and also Eqs. (2.2) and (1.3), we can write Eq. (1.2) for the determination of x_0 in the form

$$1.8\cdot 10^{-30}\left(\frac{z}{1+z}\right)^2 \xi(x_0)\,\frac{1-e^{-x_0}}{x_0^3}\,J = 1. \tag{2.10}$$

From this equation, we can determine the magnitude of the threshold photon x_0 for a given discharge current and, hence, the exact distributions of all the quantities in the discharge. To determine the temperature in the discharge, we must investigate the conditions (2.4), for which the temperature can be assumed homogeneous over the radius. For $A_0 < 2D_0$, the temperature decreases away from the axis, and if the opposite inequality is satisfied, the temperature increases toward the edges. We first consider the first of inequalities (2.4). It can easily be reduced to the form

$$7\cdot 10^{-31}\left(\frac{z}{1+z}\right)^2 \frac{\xi^2(x_0)}{f(x_0)}\,J^2 \ll 1. \tag{2.11}$$

Thus we can eliminate the discharge current J using (2.10), and we find that x_0, for a homogeneous temperature, satisfies the inequality

$$\xi(x_0) \ll 2.6\,\frac{1-e^{-x_0}}{x_0^3}\,f(x_0) = F(x_0). \tag{2.12}$$

We shall analyze this inequality. Figure 1 shows curves of the functions in the right and left parts of this inequality. We see that the region of possible values of x_0 that satisfy inequality (2.12) are included within the limits $2 \approx x_{0\,\min} < x_0 < x_{0\,\max} \approx 12$. The second of inequalities (2.4) is written in the form

$$2.2\psi(x_0) \ll F(x_0). \tag{2.13}$$

Figure 1 shows that this inequality does not place any additional constraints on x_0, since $x_{0\,\min}$, as usual, ≈ 2, $x_{0\,\max} \to \infty$. Thus, the temperature in the plasma can be assumed to be homogeneous over the cross section, with the following inequalities being satisfied:

$$2 \approx x_{0\,\min} < x_0 < x_{0\,\max} \approx 12. \tag{2.14}$$

If we now substitute the limiting values of x_0 into Eq. (2.10), then we find that the discharge current can vary over the limits (in amperes)

$$5.3\cdot 10^5\left(\frac{1+z}{z}\right) < J < 4.2\cdot 10^6\left(\frac{1+z}{z}\right). \tag{2.15}$$

In a completely opaque plasma [6], the magnitude of the discharge current is bounded by the inequalities

$$1.4\cdot 10^6\left(\frac{1+z}{z}\right) < J < 4.3\cdot 10^6\left(\frac{1+z}{z}\right). \tag{2.16}$$

Hence we see that within the limits of accuracy of the calculation, the upper limit does not change. This is natural since the upper limit of the current corresponds to $x_{0\,\max} \approx 12$, i.e., near the upper limit, the plasma is opaque practically over the entire spectral range. At the same time, the lower limit of the discharge current is strongly displaced toward lower values. This is a consequence of the fact that the temperature is homogeneous in the discharge, i.e., the magnitude of radiant thermal conductivity must be capable of guaranteeing the presence of long-wave photons in the range from 0 to x_0.

We now consider the condition $A_0 < 2D_0$, which corresponds to a decrease in temperature from the discharge axis to the periphery. We can easily show that it reduces to the form

$$\zeta(x_0) = \xi(x_0)\,\frac{1-e^{-x_0}}{x_0^3} > 2e^{-x_0}. \tag{2.17}$$

As follows from Fig. 2, this expression is violated only for $x_0 \leqslant 2$, i.e., outside of the region under consideration.

In conclusion, we comment on the total radiation output of a gray discharge. The ratio of the total radiation flux of an opaque discharge S* to the radiation flux of a gray discharge S is given by the following expression:

$$\eta = \frac{1}{\xi(x_0)} \int_0^\infty \frac{x^3}{e^x - 1} dx \approx \frac{6.5}{\xi(x_0)}. \tag{2.18}$$

Hence we see that this ratio depends on x_0, i.e., on the strength of the discharge current. For large x_0, i.e., for values of current close to J_{max}, $\eta \to 1$, and the gray discharge behaves as an absolute black body. For small discharge currents (close to J_{min}) the quantity η is different from unity, and its maximal value equals

$$\eta_{max} = \frac{6.5}{\xi(x_{0\,min})} \approx 3. \tag{2.19}$$

This indicates that for a discharge current close to J_{min}, the efficiency of a gray discharge is greater than the efficiency of an opaque discharge – by approximately a factor of 3. Assuming z = 1, we obtain $J_{min} = 1.1 \cdot 10^6$ A; in the opaque discharge, for z = 1, we have $J_{min} = 2.8 \cdot 10^6$ A.

3. Stability of Small Perturbations in a Gray Plasma

In the case being considered, of a gray discharge with a homogeneous temperature distribution, power instabilities (constrictions and bends) should develop in entirely the same way as for the case of a discharge in an optically opaque plasma. The point is that in an investigation of power instabilities, the equation of heat balance, which contains all the characteristics of "grayness" of the plasma, generally cannot be taken into account; therefore, the entire analysis carried out in [1, 4, 5] remains valid also in the case under consideration; the maximal value of the growth rate for power instabilities is

$$\gamma_{max} \lesssim v_s / r_p. \tag{3.1}$$

Generally a superheat instability does not develop in an optically opaque plasma. We show that in a gray discharge a superheat instability resulting from the ohmic heating of a plasma is also regulated by the radiant thermal conductivity connected with the long-wave radiation.

In order to investigate the superheat instability, as is known [2, 3], it is sufficient to consider only the short-wave radial perturbations, described within the framework of the zero approximation of geometrical optics [10]. Therefore, the perturbed quantities can be represented in the form $e^{-i\omega t + i\mathbf{kr}}$, where $\mathbf{k}r_p \gg 1$. We consider the high-frequency region $\omega \gg kv_s$, since the superheat instability, if it exists, in general, by virtue of (2.1) can develop only in this region. The heat-transfer equation in the discharge can be written in the form [7]

$$\rho T \frac{1+z}{M}\left[\frac{\partial}{\partial t} + (\mathbf{v}\nabla)\right] \ln \frac{T^{3/2}}{\rho} = \sigma E^2 - \gamma_0 N^2 \sqrt{T} z^3 e^{-x_0} + \operatorname{div} \frac{\beta_0 f(x_0) T^{13/2}}{z^3 N^2} \nabla T. \tag{3.2}$$

Under the conditions

$$\frac{c^2 k^2}{4\pi\sigma_0} \gg \omega \gg kv_s, \tag{3.3}$$

the electric field during the temperature oscillations becomes smoothed out over the discharge cross section, but the density, on the contrary, does not change. Furthermore, we can neglect the second term on the left side of Eq. (3.2). Converting to small perturbations of the equilibrium temperature $T \to T_0 + T_1$ and linearizing Eq. (3.2), we obtain

$$\frac{\partial T_1}{\partial t} = \frac{2}{3P_0}\left(\frac{j_0^2}{\sigma_0} - k^2 C_0 T_0\right) T_1. \tag{3.4}$$

Hence we obtain

$$\omega = i \frac{2}{3P_0}\left(\frac{j_0^2}{\sigma_0} - k^2 C_0 T_0\right). \tag{3.5}$$

This equation can also be obtained from a more rigorous consideration, i.e., from the eikonal equation; the linearized system of equations of motion (1.1) reduces to the eikonal equation in the limit of short oscillation wavelengths (over the radius), described in the framework of the zero approximation of geometrical optics. Note that the first term in (3.5) is simply the growth rate of a high-frequency superheat instability in a transparent plasma [2, 3]. We can easily show that in a gray plasma this instability is stabilized by the long-wave radiation described by the second term in (3.5), and practically does not develop, since both terms in (3.5) are of the same order, and the condition $\omega \gg kv_s$ is not satisfied.

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